113 Class Problems: Finitely Generated Groups

1. Consider the group $(\mathbb{C} \setminus \{0\}, \times)$.

(a) Prove that for any $m \in \mathbb{N}$, there exists a cyclic subgroup of $\mathbb{C} \setminus \{0\}$ of size m.

(b) Prove that this subgroup is the unique such subgroup.

(c) List all possible single set generators for this subgroup.

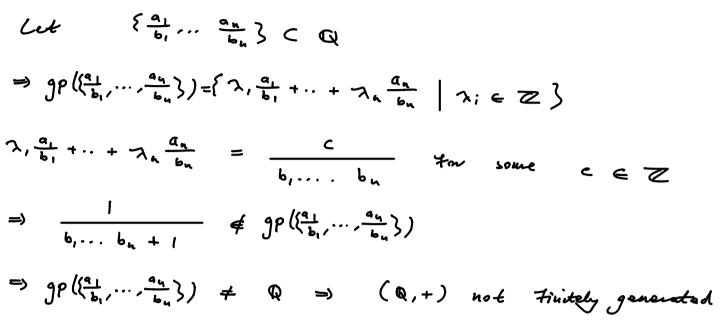
Solution:

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Assume $\exists x \in \mathbb{C} \setminus \{o\}$ such that $|gp(\{z,z\})| = m$ $(=) \quad \text{srd}(x) = m \quad (=) \quad x^{m} = l \quad \text{and} \quad x^{d} \neq l \quad \dot{f} \quad 0 < d < m$ reil = 1 (=) r=1 and 0 = 271k where ker If $x = re^{i\theta}$ and $x = 1 = r^{m}e^{im\theta} = 1 = r = 1$ and $m\theta = e\pi e$ =) x = e^{i <u>em</u> tor k e Z} Observe $\left(e^{i\frac{2\pi}{m}}\right)^{d} = e^{i\frac{2\pi d}{m}} \neq 1$ if 0 < d < m, here $\operatorname{ord}\left(e^{i\frac{\varepsilon\pi}{m}}\right) = m \implies \left|gp\left(\frac{\varepsilon\pi}{e^{m}}\right)\right| = m \left(gp\left(\frac{\varepsilon\pi}{e^{m}}\right)\right) = \left\{e^{\frac{\varepsilon\pi}{m}} \mid \varepsilon \in \mathbb{Z}\right\}$ $\{1, e^{\frac{2\pi i}{2n}}, e^{\frac{2\pi i}{2n}}, \dots, e^{\frac{2\pi i}{2n}}, \dots, e^{\frac{2\pi i}{2n}}, \dots, u^{\frac{2\pi i}{2n}}\}$ b) If HCC(50) is a cyclic subgroup, |H|=m Ventius of regular =) $\exists y \in H$ such that $y^{m} = 1$ and $gp(\xi_{y}) = H$ m-gon antered at nigin $\mathcal{Y} \in \mathcal{G}p(\{e^{\frac{2\pi i}{m}}\}) \Rightarrow \mathcal{H} \subset \mathcal{G}p(\{e^{\frac{2\pi i}{m}}\}) \Rightarrow \mathcal{H} = \mathcal{G}p(\{e^{\frac{2\pi i}{m}}\})$ =) Let $e^{\frac{2\pi i k}{m}} \in gp(e^{\frac{2\pi i}{m}})$ be a generator c) $\iff \left(e^{\frac{2\pi i k}{m}}\right)^{d} \neq \mathcal{V} \quad \forall \quad o < d < m$ (3) kd & Z V ocd < m ⇔

H(F(k,m) =), ie $e^{\frac{\pi i k}{m}}$ is a generation (=) k / m coprime

2. Prove that (Q, +) is not finitely generated.Solution:



3. Let (G, *) be a group and $x \in G$ such that $G = gp(\{x\})$. Let H be a second group and $\phi, \psi : G \to H$ be two homomorphisms. Recall that $\phi = \psi$ means that $\phi(g) = \psi(g)$ for all $g \in G$. Prove the following:

$$\phi = \psi \iff \phi(x) = \psi(x).$$

Solution:
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